Systems of Linear Equations

Nooshin Maghsoodi

Noshirvani University

These slides are adapted from Linear Algebra course in UESTC

Outline

A **linear equation** in the **variables** $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1, \ldots, a_n$ are real or complex numbers.

A **linear equation** in the **variables** $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1, \ldots, a_n$ are real or complex numbers.

Examples

Linear Equations:

$$3x_1 - 5 = x_2 \quad \text{and} \quad \sqrt{2}x_1 = 3 + x_3$$

A **linear equation** in the **variables** $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1, \ldots, a_n$ are real or complex numbers.

Examples

Linear Equations:

$$3x_1 - 5 = x_2 \quad \text{and} \quad \sqrt{2}x_1 = 3 + x_3$$

Not Linear Equations:

$$2x_1 - x_1x_2 = 2 \quad \text{and} \quad \sqrt{x_1} = 3 + x_3$$

Linear System

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables - say, $x_1, \ldots, x_n$.

Linear System

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables - say, $x_1, \ldots, x_n$.

An example

$$\begin{array}{llll} x_1 & +x_2 & -x_3 & = 1.5 \\ 2x_1 & & +2x_3 & = -1 \end{array}$$

Solution of Linear System

Solution of Linear System

A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the variables $x_1, \ldots, x_n$ are substituted by $s_1, \ldots, s_n$.

Solution of Linear System

Solution of Linear System

A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the variables $x_1, \ldots, x_n$ are substituted by $s_1, \ldots, s_n$.

$(0.5, 0, -1)$ is a solution of the previous system.

Solution of Linear System

- The set of all possible solutions is called the **solution set** of the linear system.
- Two linear systems are called **equivalent** if they have the same solution set.

Situations for Solution Set

Example

Find the solution of the following linear system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

Situations for Solution Set

Example

Find the solution of the following linear system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

It amounts to finding the **intersection to two lines.**

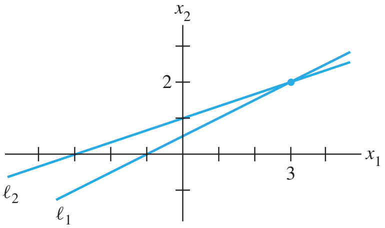
Situations for Solution Set

Example

Find the solution of the following linear system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

It amounts to finding the **intersection to two lines.**



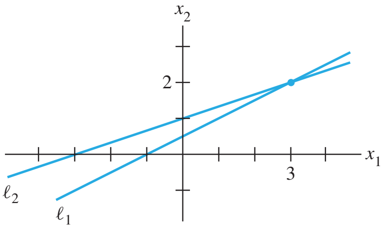
Situations for Solution Set

Example

Find the solution of the following linear system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

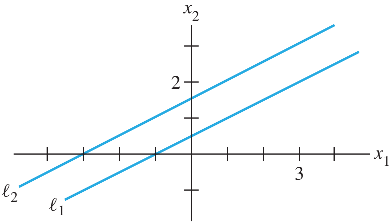
It amounts to finding the **intersection to two lines.**



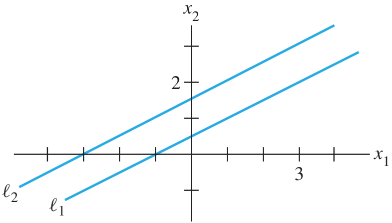
A Unique Solution

$$
\begin{aligned}
x_1 - 2x_2 &= -1 \\
-x_1 + 2x_2 &= 3
\end{aligned}
$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$



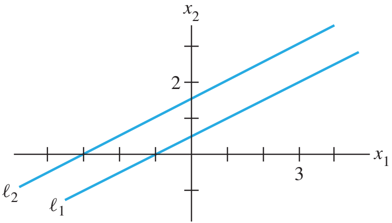
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$



No Solution

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

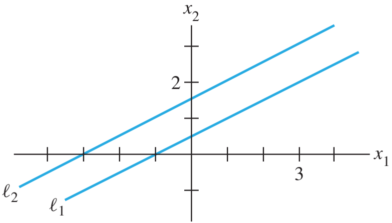
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



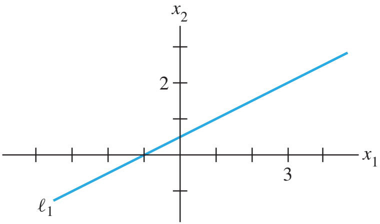
No Solution

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

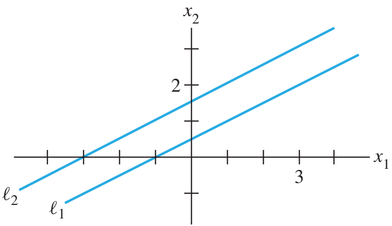
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



No Solution

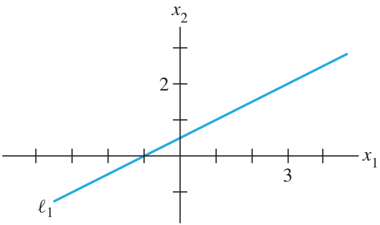


$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$



No Solution

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



Infinitely Many Solutions

Three Possible Situations

For a given linear system, it could

Three Possible Situations

For a given linear system, it could

- Have a unique solution.

Three Possible Situations

For a given linear system, it could

- Have a unique solution.
- Have infinitely many solutions.

Three Possible Situations

For a given linear system, it could

- Have a unique solution.
- Have infinitely many solutions.
- Have no solution.

Three Possible Situations

For a given linear system, it could

- Have a unique solution.
- Have infinitely many solutions.
- Have no solution.

For the first two situations, the corresponding linear system is **consistent**, *i.e.* it has at least one solution; otherwise the linear system is **inconsistent**.

Existence and Uniqueness Questions

Two fundamental questions about a linear system

Existence and Uniqueness Questions

Two fundamental questions about a linear system

1. **Existence**: Is the system consistent; that is, does at least one solution exist?

Existence and Uniqueness Questions

Two fundamental questions about a linear system

1. **Existence**: Is the system consistent; that is, does at least one solution exist?
2. **Uniqueness**: If a solution exists, is it the only one; that is, is the solution unique?

Existence and Uniqueness Questions

Two fundamental questions about a linear system

1. **Existence**: Is the system consistent; that is, does at least one solution exist?
2. **Uniqueness**: If a solution exists, is it the only one; that is, is the solution unique?

- How to answer the two questions?

Existence and Uniqueness Questions

Two fundamental questions about a linear system

1. **Existence**: Is the system consistent; that is, does at least one solution exist?
2. **Uniqueness**: If a solution exists, is it the only one; that is, is the solution unique?

- How to answer the two questions?
- Solving a linear system.

Existence and Uniqueness Questions

Two fundamental questions about a linear system

1. **Existence**: Is the system consistent; that is, does at least one solution exist?
2. **Uniqueness**: If a solution exists, is it the only one; that is, is the solution unique?

- How to answer the two questions?
- Solving a linear system.
- But before that . . . .

Matrix Notation

Definition

A **matrix** is an array of real or complex numbers.

Matrix Notation

Definition

A **matrix** is an array of real or complex numbers.

It is a convention to denote a matrix by an **upper case letter**, for example

$$A = \begin{bmatrix} -2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Matrix Notation

Definition

A **matrix** is an array of real or complex numbers.

It is a convention to denote a matrix by an **upper case letter**, for example

$$A = \begin{bmatrix} -2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

- The size of a matrix refers to the number of **rows** and number of **columns**.

Matrix Notation

Definition

A **matrix** is an array of real or complex numbers.

It is a convention to denote a matrix by an **upper case letter**, for example

$$A = \begin{bmatrix} -2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

- The size of a matrix refers to the number of **rows** and number of **columns**.
- The size of the above matrix is 2×3, reads *2-by-3*.

A linear system can be recorded compactly in a rectangular matrix.

A linear system can be recorded compactly in a rectangular matrix.

$$\begin{array}{rrrr} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{array}$$

A linear system can be recorded compactly in a rectangular matrix.

$$\begin{aligned} x_1 \quad -2x_2 \quad +x_3 &= 0 \\ 2x_2 \quad -8x_3 &= 8 \\ 5x_1 \qquad\qquad -5x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

A linear system can be recorded compactly in a rectangular matrix.

$$\begin{array}{rrrl} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

- On the left, the **coefficient matrix**,
- On the right, the **augmented matrix**.

Practice Problems

1. Write down the coefficient matrix and augmented matrix of the following linear system

$$\begin{aligned} x_1 + x_3 &= 10 \\ 2x_2 - 8x_3 &= 0 \\ x_1 - 2x_2 &= 3 \end{aligned}$$

2. Is (3, 4, -2) a solution of the following system?

$$\begin{array}{rrrr} 5x_1 & -x_2 & +1x_3 & = 7 \\ -2x_2 & +6x_2 & +9x_3 & = 0 \\ -7x_1 & 5x_2 & -3x_3 & = -7 \end{array}$$

Solving a Linear System

Strategy

Replace the system with an equivalent system (*with the same solution set*) that is easier to solve.

Solving a Linear System

Strategy

Replace the system with an equivalent system (*with the same solution set*) that is easier to solve.

- What kind of linear system is easy to solve?

Solving a Linear System

Strategy

Replace the system with an equivalent system (*with the same solution set*) that is easier to solve.

- What kind of linear system is easy to solve?
- Let us see an Example

Solving a Linear System

$$\begin{aligned} x_1 \quad -2x_2 \quad +x_3 &= 0 \\ 2x_2 \quad -8x_3 &= 8 \\ 5x_1 \quad -5x_3 &= 10 \end{aligned}$$

Solving a Linear System

$$\begin{aligned} x_1 \quad -2x_2 \quad +x_3 &= 0 \\ 2x_2 \quad -8x_3 &= 8 \\ 5x_1 \qquad\qquad -5x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Solving a Linear System

$$\begin{aligned} x_1 \quad -2x_2 \quad +x_3 &= 0 \\ 2x_2 \quad -8x_3 &= 8 \\ 5x_1 \qquad\qquad -5x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

$$\begin{array}{r} -5 \cdot \text{[equation 1]} \\ + \text{[equation 3]} \\ \hline \text{[new equation 3]} \end{array} \qquad \begin{array}{rcrcrcr} -5x_1 & + & 10x_2 & - & 5x_3 & = & 0 \\ 5x_1 & & & - & 5x_3 & = & 10 \\ \hline & & 10x_2 & - & 10x_3 & = & 10 \end{array}$$

The third equation is replaced (replacement)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

The third equation is replaced (replacement)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Then the second equation is scaled (scaling)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Another replacement

$$\begin{array}{r} -10 \cdot [\text{equation } 2] \\ + [\text{equation } 3] \\ \hline [\text{new equation } 3] \end{array} \qquad \begin{array}{rcrcr} -10x_2 & + & 40x_3 & = & -40 \\ 10x_2 & - & 10x_3 & = & 10 \\ \hline & & 30x_3 & = & -30 \end{array}$$

Another replacement

$$\begin{array}{r} -10 \cdot \text{[equation 2]} \\ + \text{[equation 3]} \\ \hline \text{[new equation 3]} \end{array} \qquad \begin{array}{rcr} -10x_2 + 40x_3 &=& -40 \\ 10x_2 - 10x_3 &=& 10 \\ \hline 30x_3 &=& -30 \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & 30x_3 & = & -30 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Another replacement

$$\begin{array}{rrcrcr} -10\cdot\text{[equation 2]} & -10x_2 & + & 40x_3 & = & -40 \\ +\text{[equation 3]} & 10x_2 & - & 10x_3 & = & 10 \\ \hline \text{[new equation 3]} & & & 30x_3 & = & -30 \end{array}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Another scaling

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Backward Substitution

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Backward Substitution

$$\begin{aligned} x_1 - 2x_2 + \quad x_3 &= \quad 0 \\ x_2 - 4x_3 &= \quad 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

1. From the third equation, we have $x_3 = -1$

Backward Substitution

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

1. From the third equation, we have $x_3 = -1$
2. With $x_3 = -1$ substituted in the second equation, we can have $x_2 = 0$ by doing simple algebra

Backward Substitution

$$\begin{aligned} x_1 - 2x_2 + \quad x_3 &= \quad 0 \\ x_2 - 4x_3 &= \quad 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

1. From the third equation, we have $x_3 = -1$
2. With $x_3 = -1$ substituted in the second equation, we can have $x_2 = 0$ by doing simple algebra
3. With $x_2 = 0$ and $x_3 = -1$ substituted in the first equation, we have $x_1 = 1$

Upper triangular linear system is easy to solve.

Upper triangular linear system is easy to solve.

- Transform a linear system to an upper triangular linear system via:

Upper triangular linear system is easy to solve.

- Transform a linear system to an upper triangular linear system via:

Elementary Row Operations

- Replacement: Replace one row by the sum of itself and a multiple of another row

Upper triangular linear system is easy to solve.

- Transform a linear system to an upper triangular linear system via:

Elementary Row Operations

- Replacement: Replace one row by the sum of itself and a multiple of another row
- Scaling: Multiply all the entries in a row by a nonzero constant

Upper triangular linear system is easy to solve.

- Transform a linear system to an upper triangular linear system via:

Elementary Row Operations

- Replacement: Replace one row by the sum of itself and a multiple of another row
- Scaling: Multiply all the entries in a row by a nonzero constant
- Row Interchange: Interchange two rows

Elementary Row Operations

- Row equivalent: if there is a sequence of elementary operation that transforms one matrix into the other.

Elementary Row Operations

- Row equivalent: if there is a sequence of elementary operation that transforms one matrix into the other.
-

Elementary Row Operations

- Row equivalent: if there is a sequence of elementary operation that transforms one matrix into the other.
-

Equivalent System

If the augmented matrices of two linear systems are row equivalent, then the two linear systems have the same solution set.

Example

Determine if the system is consistent:

$$
\begin{aligned}
x_2 \quad -4x_3 &= 8 \\
2x_1 \quad -3x_2 \quad +2x_3 &= 1 \\
4x_1 \quad -8x_2 \quad +12x_3 &= 1
\end{aligned}
$$

Example

Determine if the system is consistent:

$$\begin{aligned} x_2 \quad -4x_3 &= 8 \\ 2x_1 \quad -3x_2 \quad +2x_3 &= 1 \\ 4x_1 \quad -8x_2 \quad +12x_3 &= 1 \end{aligned}$$

Solution: The augmented matrix is

Example

Determine if the system is consistent:

$$\begin{array}{rrrl} & x_2 & -4x_3 & = 8 \\ 2x_1 & -3x_2 & +2x_3 & = 1 \\ 4x_1 & -8x_2 & +12x_3 & = 1 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

Example

Determine if the system is consistent:

$$\begin{array}{rrrl} & x_2 & -4x_3 & = 8 \\ 2x_1 & -3x_2 & +2x_3 & = 1 \\ 4x_1 & -8x_2 & +12x_3 & = 1 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

Interchange rows 1 and 2:

Example

Determine if the system is consistent:

$$\begin{array}{rrrl} & x_2 & -4x_3 & = 8 \\ 2x_1 & -3x_2 & +2x_3 & = 1 \\ 4x_1 & -8x_2 & +12x_3 & = 1 \end{array}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

Interchange rows 1 and 2:

Practice Problems

1. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent. If yes, find the solution.

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$