Introduction to Determinants

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These slides are adapted from Linear Algebra course in UESTC

Outline

1. Determinants
2. Cofactor expansion

Recall

Definition

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- A 2×2 matrix is invertible if and only if its determinant is nonzero.

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For instance, if

$$A = \begin{pmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{pmatrix}$$

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$$A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Recursive Definition of Determinant

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The determinant of an $n \times n$ matrix $A = [a_{ij}$ is the sum of n terms of the form $\pm a_{1j}\text{det}A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \cdots, a_{1n}$ are from the first row of A. In symbols,

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$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

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$$\begin{aligned} \text{det} A &= a_{11} \text{det} A_{11} - a_{12} \text{det} A_{12} + \cdots \\ &\qquad + (-1)^{1+n} a_{1n} \text{det} A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \text{det} A_{1j} \end{aligned}$$

Example

Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$\det A = 1 \cdot \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$
$$= 1(0-2) - 5(0-0) + 0(-4-0) = -2$$

Cofactor

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Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n},$$

which is a **cofactor expansion across the first row** of A.

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The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example

Use a cofactor expansion to compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\begin{aligned}\det A &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= (-1)^{3+1}a_{31}\det A_{31} + (-1)^{3+2}a_{32}\det A_{32} + (-1)^{3+3}a_{33}\det A_{33} \\ &= 0\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 0 + 2(-1) + 0 = -2\end{aligned}$$

Determinant of A Triangular Matrix

Example

Compute $\det A$, where

$$A = \begin{bmatrix} 3 & -5 & 0 & 9 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\det A = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0 \cdot C_{21} + 0 \cdot C_{31} + 0 \cdot C_{41} + 0 \cdot C_{51}$$

$$\det A = 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$\det A = 3 \cdot 2 \cdot (-2) = -12.$$

Determinant of A Triangular Matrix

Theorem

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

In-Class Exercises

- Compute the following determinant by cofactor expansion

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} \qquad \begin{vmatrix} 3 & 0 & 0 & 0 \\ -8 & 2 & 0 & 0 \\ -9 & 45 & -1 & 0 \\ 3 & 5 & 6 & 2 \end{vmatrix}$$

- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Write $4A$. Is $\det 4A = 4 \det A$?

- How may multiplication is required to calculate an $n \times n$ determinate by cofactor expansion? What does the result tell you?