3.2 Properties of Determinants

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These slides are adapted from Linear Algebra course in UESTC

Outline

1. Elementary Row Operations
2. Determinant of Transpose
3. Multiplicative and Linearity Properties

Elementary Row Operations

Exercise

Calculate the determinants of the following matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix} \quad \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

Determinants of Elementary Matrices

Exercise

Calculate the determinants of the following matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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3. If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

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Compute $\det A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

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$$= - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

Exercise

Compute $\det A$, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges (without scaling).

$$\det A = (-1)^r \det U$$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} \qquad U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det U \neq 0$ $\qquad$ $\det U = 0$

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

Combination – An Exercise

Compute $\det A$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$

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SOLUTION Add row 2 to row 4

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

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$$= (-2)(1) \begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix} = -2(15) = -30$$

Determinant of Transpose

Theorem

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Multiplicative Property

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If A and B are $n \times n$ matrices, then $\det AB = \det A \det B$.

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Verify this theorem for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

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Warning

In general, $\det(A + B) \neq \det A + \det B$.

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Suppose that the j-th column of A is allowed to vary, and write

$$A = [a_1 \ \cdots \ a_{j-1} \ x \ a_{j+1} \ \cdots \ a_n]$$

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- $\mathrm{T(cX) = cT(x)}$

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- T(cx) $=$ cT(x)
- T(u+v) $=$ T(u) $+$ T(v)

Exercise

1. Let $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, find the determinants
   - $\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix}$
   - $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$

Exercise

2. Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ in as few steps as possible.
3. Let A be an $n \times n$ matrix such that $A^2 = I$. Show that $\det A = \pm 1$.
4. Show that if A is invertible, then $\det A^{-1} = \frac{1}{\det A}$

Exercise

5. Compute $$\begin{vmatrix} a+b & a & a & \cdots & a \\ a & a+b & a & \cdots & a \\ a & a & a+b & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & a+b \end{vmatrix}$$