Cramer's Rule

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These slides are adapted from Linear Algebra course in UESTC

Outline

1. Cramer's rule
2. Determinant as Area or Volume

Definition

For any $n \times n$ matrix A and any b in $\mathbb{R}^n$, let

$$A_i(b) = \begin{bmatrix} a_1 & \cdots & b & \cdots & a_n \end{bmatrix}$$

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Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, 2, \cdots, n$$

Example

Use Cramer's rule to solve the system

$$\begin{aligned} 3x_1 \quad -2x_2 &= 6 \\ -5x_1 \quad +4x_2 &= 8 \end{aligned}$$

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$$x_1 = \frac{\det A_1(b)}{\det A} = 20$$

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$$x_1 = \frac{\det A_1(b)}{\det A} = 20$$

$$x_2 = \frac{\det A_2(b)}{\det A} = 27$$

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$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

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A Formula for A^{-1}

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \text{adj} A$$

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

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- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $\det A$.