Vector Spaces and Subspaces

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These slides are adapted from Linear Algebra course in UESTC

Outline

1. Vector Space
2. Subspace
3. A Subspace Spanned by a Set

Vector Space Definition

A **vector space** is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ in V and for all scalars c and d.

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1. The sum of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u} + \boldsymbol{v}$, is in V.
2. The scalar multiple of $\boldsymbol{u}$ by c, denoted by $c\boldsymbol{u}$, is in V.

Vector Space Definition

3. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
4. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.
5. There is a zero vector $\mathbf{0}$ in V such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$.
6. For each $\boldsymbol{u}$ in V, there is a vector $-\boldsymbol{u}$ in V such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$.

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7. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$.
8. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$.
9. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$.
10. $1\boldsymbol{u} = \boldsymbol{u}$.

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- For each $\boldsymbol{u} \in V$ and scalar c
  - $0\boldsymbol{u} = 0$
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  - $-\boldsymbol{u} = (-1)\boldsymbol{u}$

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3. For $n \geq 0$, the set $\mathbb{P}_n$ of polynomials of degree at most n consists of all polynomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

with $a_0, a_1, \ldots, a_n$ coefficients and t variable.

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5. The set of all the $m \times n$ real matrices, denoted by $\mathbb{R}^{m \times n}$

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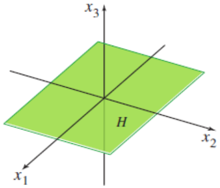

Subspaces of $\mathbb{R}^3$?

A subspace spanned by A Set

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Recall: Linear Combination and *Span*

- A **linear combination** refers to any sum of scalar multiples of vectors.
- Span$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$.

Example

Given $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in a vector space V, let $H = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. Show that H is a subspace of V.

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SOLUTION Show H satisfies the three properties.

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If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are in a vector space V, then Span$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is a subspace of V. We call **the subspace spanned (generated)** by $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$. Given any subspace H of V, a **spanning (generating) set** for H is a set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in H such that $H =$ Span$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$.

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SOLUTION Write the vectors in H as column vectors.

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Example

Is $b = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 2 \end{bmatrix}$ in Span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$,

where $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$?

Example

For what value(s) of h will y be in the subspace of $\mathbb{R}^3$ spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, if

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} -3 \\ 1 \\ -0 \end{bmatrix},$$

and $y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$.

Exercises

a) Are the following subspaces?
- The set of all the points inside and on the unit circle.
- $\begin{bmatrix} 3a+2b \\ b-1 \\ a-b \end{bmatrix}$, with $a, b \in \mathbb{R}$
- $\begin{bmatrix} a-b \\ b+c \\ c-a \end{bmatrix}$, with $a, b, c \in \mathbb{R}$