4.2 Null Space, Column Space, and Linear Transformations

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Outline

Null Space

Definition

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$$\text{Nul}A = \{\boldsymbol{x} : \boldsymbol{x} \text{ is in } \mathbb{R}^n \text{ and } A\boldsymbol{x} = \mathbf{0}\}$$

Example

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, and let $\boldsymbol{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$.

Determine if $\boldsymbol{u}$ belongs to the null space of A.

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SOLUTION Easy.

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Determine if $\boldsymbol{u}$ belongs to the null space of A.

SOLUTION Easy. Only need to check if $A\boldsymbol{u} = \mathbf{0}$.

Theorem

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The null space of an $m \times n$ matrix A is a **subspace of** $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\boldsymbol{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Example

Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a, b, c, d satisfy the equations $a - 2b + 5c = d$ and $c - a = b$. Show that H is a subspace of $\mathbb{R}^4$.

$$\begin{aligned} a - 2b + 5c - d &= 0 \\ -a - \ b + \ c \qquad &= 0 \end{aligned}$$

It is important that the linear equations defining the set H are homogeneous. Otherwise, the set of solutions will definitely *not* be a subspace (because the zero vector is not a solution of a nonhomogeneous system). Also, in some cases, the set of solutions could be empty.

Example

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

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SOLUTION

$$A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Write the corresponding linear system

$$\begin{array}{llllll} x_1 & -2x_2 & & -x_4 & +3x_5 & = 0 \\ & & x_3 & +2x_4 & -2x_5 & = 0 \\ & & & & 0 & = 0 \end{array}$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{\substack{\uparrow \\ \mathbf{u}}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\substack{\uparrow \\ \mathbf{w}}}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

Every linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is an element of Nul A and vice versa. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for Nul A. ■

Write the corresponding linear system

$$\begin{aligned} x_1 \quad -2x_2 \qquad\quad -x_4 \quad +3x_5 &= 0 \\ x_3 \quad +2x_4 \quad -2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

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Thus $\{u, v, w\}$ is a spanning set for Nul$A$.

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$$\text{Col}A = \{\boldsymbol{b} : \boldsymbol{b} = A\boldsymbol{x}, \forall \boldsymbol{x} \in \mathbb{R}^n\}$$

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- A vector $\boldsymbol{b}$ is in Col$A$ if and only if $A\boldsymbol{x} = \boldsymbol{b}$ has at least a solution.
- The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for each $\boldsymbol{b} \in \mathbb{R}^m$.

Example

Find a matrix A such that $W = \text{Col}A$.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

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Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

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2. If Nul$A$ is a subspace of $\mathbb{R}^k$, what is k?
3. Find a nonzero vector in Col$A$ (Nul$A$)

$$[\,A \quad \mathbf{0}\,] \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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3. Find a nonzero vector in Col$A$ (Nul$A$)
4. Let $\boldsymbol{u} = \begin{bmatrix} 3 & -2 & -1 & 0 \end{bmatrix}^T$, and determine if $\boldsymbol{u}$ is in Nul$A$. Could $\boldsymbol{u}$ be in Col$A$?

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5. Let $\boldsymbol{v} = \begin{bmatrix} 3 & -1 & 3 \end{bmatrix}^T$, and determine if $\boldsymbol{v}$ is in Col$A$. Could $\boldsymbol{u}$ be in Nul$A$?

Contrast Between Nul A and Col A for an m x n Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Linear Transformation

A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\boldsymbol{x}$ in V a unique vector $T(\boldsymbol{x})$ in W, s.t.

- $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u})+T(\boldsymbol{v}), \forall \boldsymbol{u}, \boldsymbol{v} \in V$
- $T(c\boldsymbol{u}) = cT(\boldsymbol{u}), \forall \boldsymbol{u} \in V, c \in \mathbb{R}$

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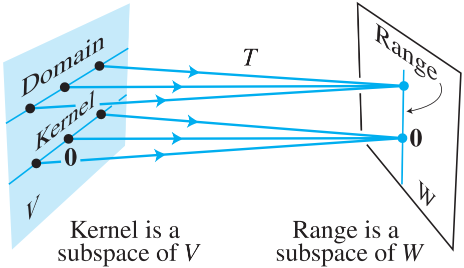
Domain
Kernel
0
V
T
Range
0
W
Kernel is a subspace of V
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Matrix Transformation

Matrix Transformation (Multiplication by Matrix)

For each x in $\mathbb{R}^n$, $T(\boldsymbol{x})$ is computed as $A\boldsymbol{x} \in \mathbb{R}^m$, where A is an $m \times n$ matrix.

- For simplicity, we denote such a matrix transformation by $\boldsymbol{x} \mapsto A\boldsymbol{x}$.

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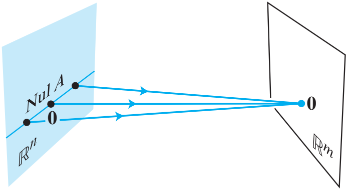
- For simplicity, we denote such a matrix transformation by $\boldsymbol{x} \mapsto A\boldsymbol{x}$.
- The matrix transformation is a linear transformation.

Example

- Nol$A$ is the kernel of the linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$



- Col$A$ is the range of the linear transformation

$$\boldsymbol{x} \mapsto A\boldsymbol{x}.$$

Exercises

1. Telling true of false
   - The null space of A is the solution set of the equation $A\boldsymbol{x} = \mathbf{0}$.
   - The column space of A is the range of the mapping $\boldsymbol{x} \mapsto A\boldsymbol{x}$.
   - If the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent, then Col$A$ is $\mathbb{R}^m$.
   - The kernel of linear transformation is a vector space.
   - Col$A$ is the set of all vectors that can be written as $A\boldsymbol{x}$ for some $\boldsymbol{x}$.
   - Col$A$ is the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$.

Exercises

2. Let $A = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if $\boldsymbol{w}$ is in Col$A$? is $\boldsymbol{w}$ in Nul$A$?
3. Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$. Show in two different ways that W is a subspace of $\mathbb{R}^3$.
4. Let A be an $n \times n$ matrix. If Col$A$=Nul$A$, show that Nul$A^2 = \mathbb{R}^n$.