4.3 Linearly Independent Sets and Bases

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These slides are adapted from Linear Algebra course in UESTC

Outline

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Linearly Independent Sets

Linearly Independent

An indexed set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_p\boldsymbol{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = \cdots = x_p = 0$.

Linearly Independent Sets

- The set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is said to be **linearly dependent** if there exist weight $c_1, \ldots, c_p$, **not all zero**, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p = 0$$

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- In the linearly dependent case, the equation defines a **linear dependence relation** among $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$.

Example

Let $\boldsymbol{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$

1. Determine if the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.
2. If possible, find a linear dependence relation among $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

$$\begin{bmatrix}1 & 4 & 2 & 0\\2 & 5 & 1 & 0\\3 & 6 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 4 & 2 & 0\\0 & -3 & -3 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix} \quad \begin{aligned} x_1 \quad - 2x_3 &= 0\\ x_2 + \quad x_3 &= 0\\ 0 &= 0\end{aligned}$$

Linear Independence of Matrix Columns

Theorem

The columns of a matrix A are linearly independent if and only if the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

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Example

Determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

are linearly independent.

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

Sets of One or Two Vectors

- A set $\{v\}$ with only one vector?
  - $v = \mathbf{0}$

Sets of One or Two Vectors

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  - $\boldsymbol{v} = \mathbf{0}$ dependent

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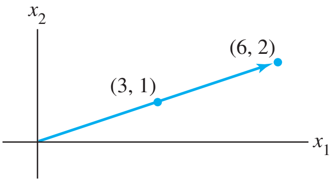
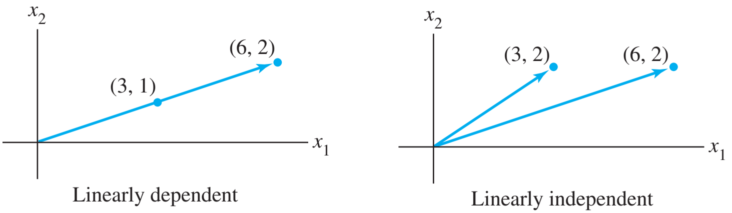
Example

Determine if the following sets of vectors are linearly independent.

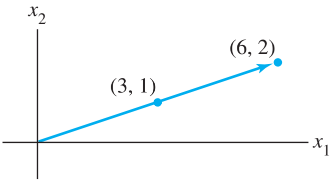
$$a).\ \boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \quad b).\ \boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$



Linearly dependent



- A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.



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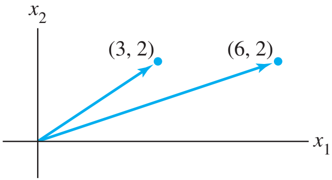

Linearly independent

- A set of two vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.
- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Sets of Two or More Vectors

Theorem

- An indexed set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ of two or more vectors is linearly dependent **if and only if** at least one of the vectors in S is a linear combination of the others.

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Theorem

- An indexed set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ of two or more vectors is linearly dependent **if and only if** at least one of the vectors in S is a linear combination of the others.
- In fact, S with $v_1 \neq 0$, is linearly dependent **if and only if** some $\boldsymbol{v}_j$ (with $j > 0$) is a linear combination of the proceeding vectors, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}$.

Example

Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$, and explain why a vector $\boldsymbol{w}$ is in Span$\{\boldsymbol{u}, \boldsymbol{v}\}$ if and only if $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly dependent.

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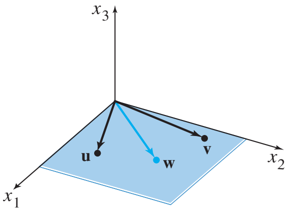

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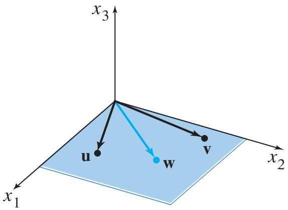

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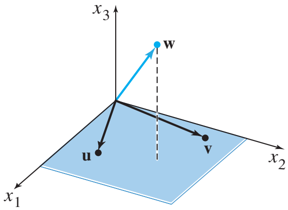

Linearly independent,
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Example

Let $p_1(t) = 1, p_2(t) = t$, and $p_3(t) = 2 - t$. Then $\{p_1, p_2, p_3\}$ is linearly dependent in $\mathbb{P}$ because $p_3 = 2p_1 - p_2$.

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$$n\overbrace{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}^{p}$$

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If a set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ contains **the zero vector**, then the set is linearly dependent.

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If a set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ contains **the zero vector**, then the set is linearly dependent.

Invertible Matrix Theorem

A is invertible if and only if the columns of A form a linearly independent set.

Outline

Basis

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_p\}$ in V is a basis for H if

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- This applies to the case when $H = V$.

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Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the columns of the $n \times n$ identity matrix I_n. That is

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The set $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is called the standard basis for $\mathbb{R}^n$.

Example

Let $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\mathbb{R}^3$.

The Spanning Set Theorem

Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ be a set in V, and let $H = \text{Span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$.

a. If one of the vectors in S -say, $\boldsymbol{v}_k$- is a linear combination of the remaining vectors in S, then the set formed from S by removing $\boldsymbol{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Outline

Bases for Nul$A$ and Col$A$

Example

Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

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SOLUTION $A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

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SOLUTION $A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Write the corresponding linear system

$$\begin{array}{rrrrrl} x_1 & -2x_2 & & -x_4 & +3x_5 & = 0 \\ & & x_3 & +2x_4 & -2x_5 & = 0 \\ & & & & 0 & = 0 \end{array}$$

Example

Find a basis for Col$B$, where

$$B = [\boldsymbol{b}_1 \cdots \boldsymbol{b}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

It can be shown that the matrix

$$A = [\boldsymbol{a}_1 \cdots \boldsymbol{a}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix B. Find a basis for Col$A$.

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The **pivot columns of a matrix** A form a basis for Col$A$.

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- The pivot columns of a matrix A are evident when A has been reduced to only echelon form.
- But, be careful, one should use the pivot columns of A itself for the basis of Col$A$.
- Row operations can change the column space of a matrix.

Two Views of a Basis

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$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix}\right\}$$

Linearly independent but does not span $\mathbb{R}^3$

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$$

A basis for $\mathbb{R}^3$

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Spans $\mathbb{R}^3$ but is linearly dependent

Exercises

1. Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 6 \end{bmatrix}$. Determine if $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis for $\mathbb{R}^3$.

Exercises

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Exercises

1) Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 6 \end{bmatrix}$. Determine if $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ a basis for $\mathbb{R}^2$?

2) Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace W spanned by $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$.

Exercises

3) Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Is $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ a basis for H?