Sec 5.1 Eigenvectors and Eigenvalues

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These slides are adapted from Linear Algebra course in UESTC

Outline

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Google's Ranking Algorithm

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Example

Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Compute Au and Av.

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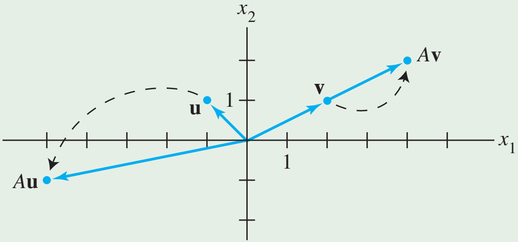

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- A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution x of $Ax = \lambda x$;
- such an x is called an **eigenvector corresponding to λ**.

Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

- Are u and v eigenvectors of A?

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Example

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- Are u and v eigenvectors of A?
- Show that 7 is an eigenvalue and find the corresponding eigenvectors.

$$A\mathbf{x} = 7\mathbf{x}$$

$$(A - 7I)\mathbf{x} = \mathbf{0}$$

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspace

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- The set of all solutions is just the null space of $A - \lambda I$.
- This set is a subspace of $\mathbb{R}^n$ and is called the **eigensapce** of A corresponding to λ.
- The eigenspace consists of **the zero** vector and all the eigenvectors corresponding to λ.

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Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

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Solution:

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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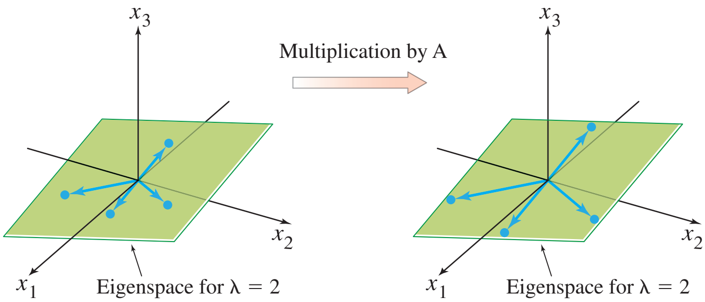
A basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.


x_3
Multiplication by A
x_3
x_2
x_1
Eigenspace for λ = 2
x_2
x_1
Eigenspace for λ = 2

Example

If $n \times n$ matrix A satisfies $A^2 = A$. Show that A only has eigenvalues 0 or 1

$$\lambda v = Av = AAv = \lambda Av = \lambda^2 v$$

Invertible Matrix Theorem (continued)

Matrix A is invertible if and only if

- 0 is not an eigenvalue of A

Theorem

If $v_1, \ldots, v_r$ are eigenvectors that correspond to **distinct** eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, \ldots, v_r\}$ is linearly independent.

Questions

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- How to find the eigenvalues of a matrix?
- Are there enough linearly independent eigenvectors to span $\mathbb{R}^n$, or to form a basis of $\mathbb{R}^n$?