Sec 5.2 The Characteristic Equation

Nooshin Maghsoodi

Noshirvani University

These slides are adapted from Linear Algebra course in UESTC

Outline

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- The secret lies in that the equation $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has nontrivial solution.
- Or $A - \lambda I$ is not invertible (singular)
- Recall the Invertible Matrix Theorem: A matrix is invertible if and only if its determinant is nonzero.

Characteristic Equation

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the **characteristic equation**

$$\det(A - \lambda I) = 0$$

Example

Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 2 & -8 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The eigenvalues of a triangular matrix are the entries on its main diagonal.

Characteristic polynomial

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- **(Algebraic) multiplicity** of an eigenvalue λ: its multiplicity as a root of the characteristic equation

Example

The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicities.

Exercises

- If $\boldsymbol{x}$ is an eigenvector of A corresponding to λ, what is $A^3\boldsymbol{x}$?
- If A is an $n \times n$ matrix and λ is an eigenvalue of A, show that 2λ is an eigenvalue of $2A$.
- Find the characteristic equation and eigenvalues of

$$A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$$

- If $A^2 = I$, show that the eigenvalues of A can only be 1 or -1

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- The complex eigvalues of a real matrix will appear in conjugate pair.
- If a real matrix A has complex eigenvalue λ, then the corresponding eigenvectors are also complex.

Eigenvalues, Trace and Determinant

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- Determinant: $\lambda_1 \cdots \lambda_n = \det A$

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- If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently $A = PBP^{-1}$.

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- Writing Q for P^{-1}, we have $Q^{-1}BQ = A$. So B is also similar to A, and we say simply that A and B **are similar**.
- Changing A into $P^{-1}AP$ is called a **similarity transformation**.

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- Are the matrices $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ similar?
- Row operations on a matrix usually change its eigenvalues.