5.3 Diagonalization

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These slides are adapted from Linear Algebra course in UESTC

Outline

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Compute A^k

Example

If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$,

Compute A^k

Example

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$D^3 = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$,

Compute A^k

Example

If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$,

$D^3 = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$, and in general

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$$

How to compute A^k?

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If $A = PDP^{-1}$ for some invertible P and D, then A^k is easy to compute.

Example

Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

Example

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$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

Solution $P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Again,

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{I}D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

In general, for $k \geq 1$,

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$$

Definition

A matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

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Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

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An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

The columns of P is called an **eigenvector basis** of $\mathbb{R}^n$

Example

Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

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Solution

- Find the eigenvalues of A.

$$\begin{aligned} 0 = \det(A - \lambda I) &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2 \end{aligned}$$

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$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution

- Find the eigenvalues of A.
- Find three linearly independent eigenvectors of A.

$$\begin{aligned} 0 = \det(A - \lambda I) &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2 \end{aligned}$$

$$\text{Basis for } \lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Example

Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution

- Find the eigenvalues of A.
- Find three linearly independent eigenvectors of A.
- Construct P, whose columns are eigenvectors.

Example

Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution

- Find the eigenvalues of A.
- Find three linearly independent eigenvectors of A.
- Construct P, whose columns are eigenvectors.
- Construct D, whose diagonal entries are eigenvalues.

Example

Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{bmatrix}$$

Example

Diagonalize the following matrix, if possible,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

The characteristic equation of A:

$$0 = |A - \lambda I| = -(\lambda - 1)(\lambda + 2)^2.$$

Basis for $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Basis for $\lambda = -2$: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Outline

Theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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Example

Determine if the following matrix is diagonalizable.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonalizable Matrix

Theorem

Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

Diagonalizable Matrix

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a. For $1 \leq k \leq p$, the **dimension of the eigenspace** for λ_k is **less than or equal to** the **multiplicity of the eigenvalue** λ_k.

Diagonalizable Matrix

Theorem

Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the **dimension of the eigenspace** for λ_k is **less than or equal to** the **multiplicity of the eigenvalue** λ_k.
b. The matrix A is diagonalizable **if and only if** the sum of the dimensions of the eigenspaces equals n,

Diagonalizable Matrix

Theorem

Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the **dimension of the eigenspace** for λ_k is **less than or equal to** the **multiplicity of the eigenvalue** λ_k.
b. The matrix A is diagonalizable **if and only if** the sum of the dimensions of the eigenspaces equals n, and this happens if and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

Diagonalizable Matrix

Theorem (continued)

- (c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the **total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$** forms an eigenvector basis for $\mathbb{R}^n$.

Exercises

- Compute A^{100}, where $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$
- Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$$