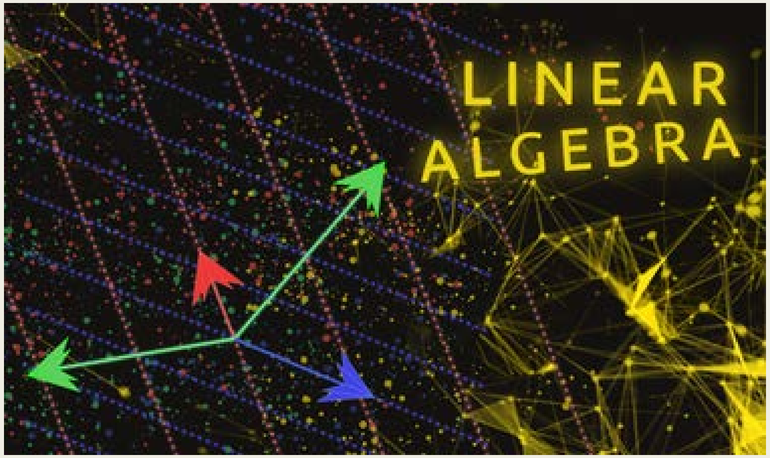

LINEAR ALGEBRA

Echelon Form Simplified

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Echelon Form

- Echelon Form of a matrix is used to solve a linear system by converting a complex matrix to a simple matrix.
- A matrix is in an **Echelon Form** if it satisfies some conditions.
- We must know how to convert a matrix into Echelon Form and simplify our matrix for further linear algebra operations.

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What is Echelon Form?

- A matrix is in an Echelon Form when it satisfies three conditions

- We will explain each of them with examples.

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What is Echelon Form?

- A definition:

  Leading entry: first non-zero element in a row

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

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What is Echelon Form?

- First condition:
  - *A row with no zero element is above all other rows.*

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

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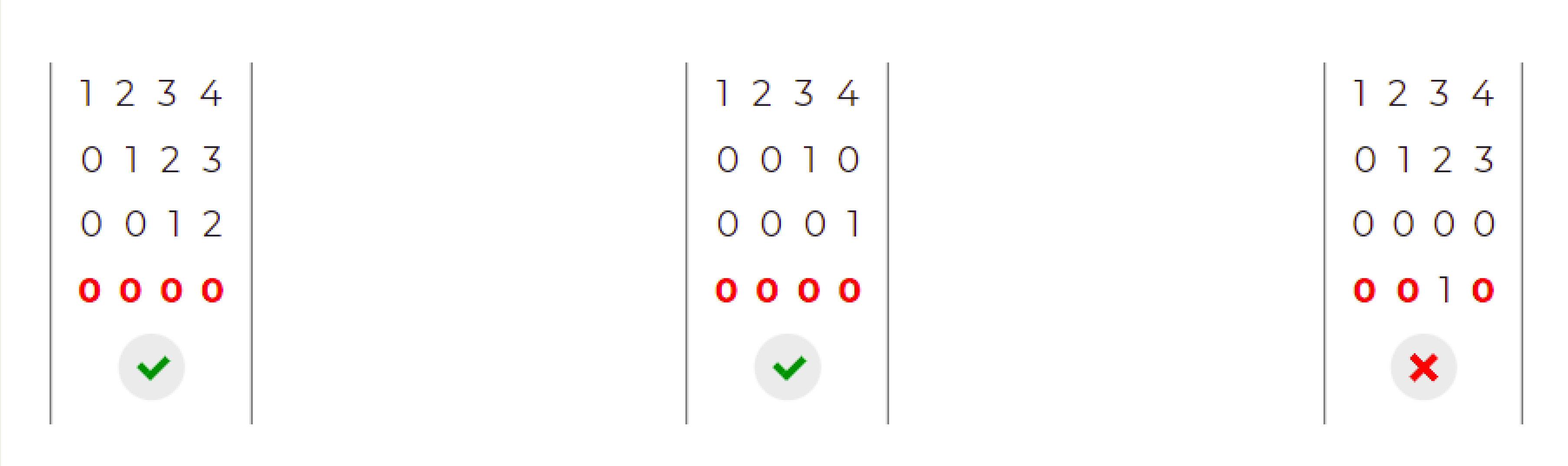
What is Echelon Form?

- Second condition:
  - *Each leading entry in a row is in the right side of the leading entry of the previous row.*

$$\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \quad ✓ \qquad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad ✓ \qquad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 5 \end{vmatrix} \quad ✗$$

What is Echelon Form?

- Third condition:
  - *A row with all zeros should be below rows having a non-zero element.*



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- IN SOME OTHER REFRENCES THE IN REF THE LEADING ENTRY IN EACH ROW SHOULD BE ONE.

- This process of converting a matrix to echelon form is known as **Gaussian elimination** process.

Reduced Row Echelon form

- This is a special form of a row echelon form matrix.
- So A row echelon form is reduced row echelon form if it satisfies the following condition:
  - *The leading entry of each row is 1.*
  - *A pivot or leading entry 1 in the row will be the only non-zero value in its columns.*
  - *So all other values in the same column will have zero value.*

$$\begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

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Solving a System Using Gaussian Elimination

- To solve a system using Gaussian elimination, we use:

  1. *Augmented matrix*
  2. *Row-echelon form*
  3. *Back-substitution*

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Solving a System Using Gaussian Elimination

1. Augmented matrix

   – *Write the augmented matrix of the system.*

2. Row-echelon form

   – *Use elementary row operations to change the augmented matrix to row-echelon form.*

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Solving a System Using Gaussian Elimination

3. Back-substitution

   - *Write the new system of equations that corresponds to the row-echelon form of the augmented matrix and solve by back-substitution.*

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E.g. 3—Solving a System Using Row-Echelon Form

- Solve the system of linear equations using Gaussian elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

  - *We first write the augmented matrix of the system.*
  - *Then, we use elementary row operations to put it in row-echelon form.*

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E.g. 3—Solving a System Using Row-Echelon Form

- $$\begin{bmatrix} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

E.g. 3—Solving a System Using Row-Echelon Form

- $$\xrightarrow[R_3+2R_1 \to R_3]{R_2-3R_1 \to R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{bmatrix}$$

E.g. 3—Solving a System Using Row-Echelon Form

- $$\xrightarrow{R_3 - 5R_2 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{10}R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

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E.g. 3—Solving a System Using Row-Echelon Form

- We now have an equivalent matrix in row-echelon form.
- The corresponding system of equations is:

$$\begin{cases} x + 2y - z = 1 \\ y + 4z = -7 \\ z = -2 \end{cases}$$

  - *We use back-substitution to solve the system.*

E.g. 3—Solving a System Using Row-Echelon Form

- $y + 4(-2) = -7$

  $y = 1$

- $x + 2(1) - (-2) = 1$

  $x = -3$

The solution of the system is:

(–3, 1, –2)

Putting in Reduced Row-Echelon Form

- If we put the augmented matrix of a linear system in reduced row-echelon form,
  then we don't need to back-substitute
  to solve the system.

  - *To put a matrix in reduced row-echelon form, we use the following steps.*
    - We see how the process might work for a 3 x 4 matrix.

Putting in Reduced Row-Echelon Form—Step 1

- Use the elementary row operations to put the matrix in row-echelon form and set each leading entry to 1.

$$\begin{bmatrix} 1 & \_ & \_ & \_ \\ 0 & 1 & \_ & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix}$$

Putting in Reduced Row-Echelon Form—Step 2

- Obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it.

$$\begin{bmatrix} 1 & \_ & \_ & \_ \\ 0 & 1 & \_ & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \_ & 0 & \_ \\ 0 & 1 & 0 & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix}$$

Putting in Reduced Row-Echelon Form—Step 2

- Begin with the last leading entry and work up.

$$\begin{bmatrix} 1 & \_ & \_ & \_ \\ 0 & 1 & \_ & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \_ & 0 & \_ \\ 0 & 1 & 0 & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \_ \\ 0 & 1 & 0 & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix}$$

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Gauss-Jordan Elimination

- Using the reduced row-echelon form to solve a system is called Gauss-Jordan elimination.
  - *We illustrate this process in the next example.*

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E.g. 4—Solving Using Reduced Row-Echelon Form

- Solve the system of linear equations, using Gauss-Jordan elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

  - *In the previous Example, we used Gaussian elimination on the augmented matrix of this system to arrive at an equivalent matrix in row-echelon form.*

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E.g. 4—Solving Using Reduced Row-Echelon Form

- We continue using elementary row operations on the last matrix in Example 3 to arrive at an equivalent matrix in reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

E.g. 4—Solving Using Reduced Row-Echelon Form

- $$\xrightarrow[R_1+R_3\to R_1]{R_2-4R_3\to R_2}\begin{bmatrix}1 & 2 & 0 & -1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -2\end{bmatrix}$$

$$\xrightarrow{R_1-2R_2\to R_1}\begin{bmatrix}1 & 0 & 0 & -3\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -2\end{bmatrix}$$

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E.g. 4—Solving Using Reduced Row-Echelon Form

- We now have an equivalent matrix in reduced row-echelon form.
- The corresponding system of equations is:

$$\begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

  - *Hence, we immediately arrive at the solution (–3, 1, –2).*

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