6.1 Inner Product, Length, and Orthognality

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These slides are adapted from Linear Algebra course in UESTC

Outline

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Purpose

- We want to extend the geometric concepts of length, distance and perpendicularity, defined on $\mathbb{R}^2$ and $\mathbb{R}^3$, to $\mathbb{R}^n$.

Inner Product

- If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, we regard them as $n \times 1$ matrices, then $\boldsymbol{u}^T \boldsymbol{v}$ is a scalar.

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If

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

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$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i$$

Example

Compute $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{v} \cdot \boldsymbol{u}$ for $\boldsymbol{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

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- Generalization?

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- A vector whose length is 1 is called a **unit vector**.
- **Normalizing** – If we divide a nonzero vector v by its length, we obtain a unit vector u.

Properties of Length

- positivity

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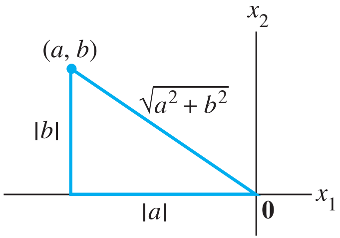

Example

Let $\boldsymbol{v} = \begin{bmatrix} 1 & -2 & 2 & 0 \end{bmatrix}^T$. Find a unit vector $\boldsymbol{u}$ in the same direction as $\boldsymbol{v}$

Distance

For $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$, the **distance between $\boldsymbol{u}$ and $\boldsymbol{v}$**, written as $\text{dist}(\boldsymbol{u}, \boldsymbol{v})$, is the length of the vector $\boldsymbol{u} - \boldsymbol{v}$.

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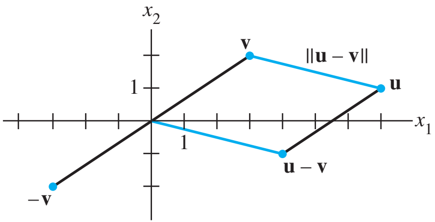
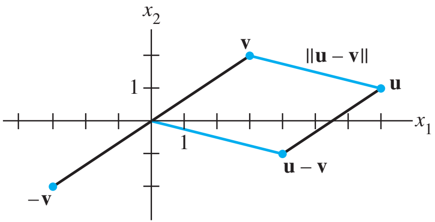
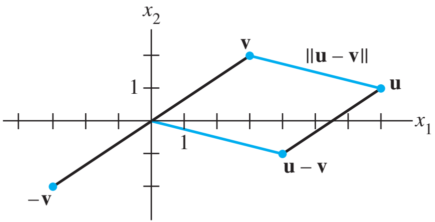

x_2
$\mathbf{v}$
$\|\mathbf{u} - \mathbf{v}\|$
$\mathbf{u}$
1
x_1
1
$\mathbf{u} - \mathbf{v}$
$-\mathbf{v}$



Example

Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$, then

$$\text{dist}(\boldsymbol{u}, \boldsymbol{v})$$



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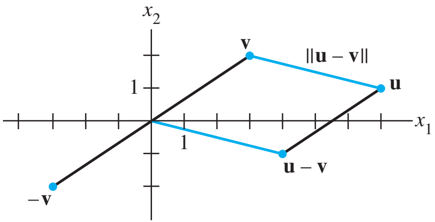

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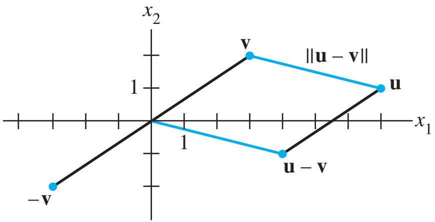
$$\text{dist}(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v})}$$



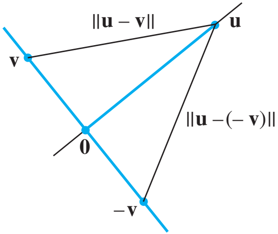
Example

Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned} \text{dist}(\boldsymbol{u}, \boldsymbol{v}) &= \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v})} \\ &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2} \end{aligned}$$

Outline

Orthogonal Vectors



Consider $\mathbb{R}^2$ or $\mathbb{R}^3$. Two lines (vectors) are **geometrically perpendicular** if and only if the distance from $\boldsymbol{u}$ to $\boldsymbol{v}$ is the same as the distance from $\boldsymbol{u}$ to $-\boldsymbol{v}$.

Definition

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

$$\begin{aligned}
[\,\mathrm{dist}(\mathbf{u}, -\mathbf{v})\,]^2 &= \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \\
&= (\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} + \mathbf{v}) \\
&= \mathbf{u}\cdot(\mathbf{u} + \mathbf{v}) + \mathbf{v}\cdot(\mathbf{u} + \mathbf{v}) && \text{Theorem 1(b)} \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} && \text{Theorem 1(a), (b)} \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v} && \text{Theorem 1(a)}
\end{aligned}$$

$$\begin{aligned}
[\mathrm{dist}(\mathbf{u}, \mathbf{v})]^2 &= \|\mathbf{u}\|^2 + \|-\mathbf{v}\|^2 + 2\mathbf{u}\cdot(-\mathbf{v}) \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}
\end{aligned}$$

Definition

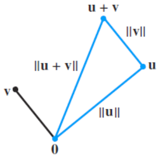
Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

- Zero vector is orthogonal to every vector.

The Pythagorean Theorem

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if and only if

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$



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- If a vector $\boldsymbol{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\boldsymbol{z}$ is said to be **orthogonal to W**.
- The set of all vectors that are orthogonal to W is called the **orthogonal complement** of W, and is denoted by $W^\perp$.

Properties of $W^\perp$

- A vector x is in $W^\perp$ if and only if x is orthogonal to every vector in a set that spans W.

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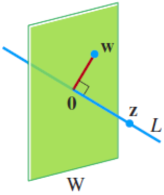
- $W^\perp$ is a subspace of $\mathbb{R}^n$.



Theorem

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row}A)^{\perp} = \text{Nul}A \quad \text{and} \quad (\text{Col}A)^{\perp} = \text{Nul}A^T$$

Exercises

- Find a unit vector in the direction of the given vector.

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$$

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- Find the distance between $\boldsymbol{x} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

Exercises

- Telling True or False.
  - $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2$.
  - For a square matrix A, vectors in Col$A$ are orthogonal to vectors in Nul$A$.
  - If vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p$ span a subspace W and if $\boldsymbol{x}$ is orthogonal to each $\boldsymbol{v}_j$ for $j = 1, \cdots, p$, then $\boldsymbol{x}$ is in $W^{\perp}$.