6.2 Orthogonal Sets

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These slides are adapted from Linear Algebra course in UESTC

Outline

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Orthogonal Set

Definition

A set of vectors $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, that is, if $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$ whenever $i \neq j$.

Example

Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal set, where

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

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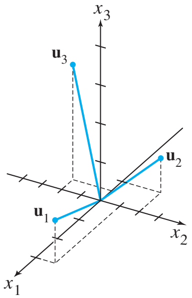

Theorem

If $S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is **linearly independent** and hence is a basis for the subspace spanned by S.

PROOF If $\mathbf{0} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$ for some scalars $c_1, \ldots, c_p$, then

$$\begin{aligned} 0 = \mathbf{0}\cdot\mathbf{u}_1 &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{u}_1 \\ &= (c_1\mathbf{u}_1)\cdot\mathbf{u}_1 + (c_2\mathbf{u}_2)\cdot\mathbf{u}_1 + \cdots + (c_p\mathbf{u}_p)\cdot\mathbf{u}_1 \\ &= c_1(\mathbf{u}_1\cdot\mathbf{u}_1) + c_2(\mathbf{u}_2\cdot\mathbf{u}_1) + \cdots + c_p(\mathbf{u}_p\cdot\mathbf{u}_1) \\ &= c_1(\mathbf{u}_1\cdot\mathbf{u}_1) \end{aligned}$$

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- An orthogonal basis is much nicer than other basis.

Theorem

Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$ be an **orthogonal basis** for a subspace W of $\mathbb{R}^n$. For each $\boldsymbol{y} \in W$, the weights in the linear combination

$$\boldsymbol{y} = c_1\boldsymbol{u}_1 + \cdots + c_p\boldsymbol{u}_p$$

are given by

$$c_j = \frac{\boldsymbol{y} \cdot \boldsymbol{u}_j}{\boldsymbol{u}_j \cdot \boldsymbol{u}_j} \quad (j = 1, \ldots, p)$$

Example

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } \boldsymbol{x} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$$

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Problem of Orthogonal Decomposition

Given a nonzero vector $\boldsymbol{u} \in \mathbb{R}^n$, consider the problem of decomposing a vector $\boldsymbol{y} \in \mathbb{R}^n$ into the sum of two vectors, one a multiple of $\boldsymbol{u}$ and the other **orthogonal** to $\boldsymbol{u}$.

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$$\boldsymbol{y} = \hat{\boldsymbol{y}} + \boldsymbol{z}$$

where $\hat{\boldsymbol{y}} = \alpha \boldsymbol{u}$ for some scalar α and $\boldsymbol{z}$ is some vector orthogonal to $\boldsymbol{u}$.

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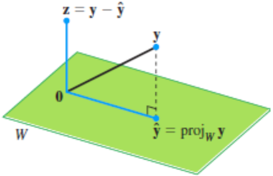
- Sometimes $\hat{y}$ is denoted by $\text{proj}_L y$ and is called the **orthogonal projection of y onto L**.

$$\hat{y} = \text{proj}_L y = \frac{y \cdot u}{u \cdot u} u$$



Example

Let $\boldsymbol{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\boldsymbol{y}$ onto span$\{\boldsymbol{u}\}$.

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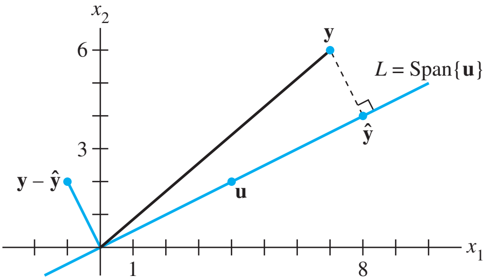
$$\mathbf{y}\cdot\mathbf{u} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 40$$

$$\mathbf{u}\cdot\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 20$$

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{40}{20}\mathbf{u} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\underset{\mathbf{y}}{\begin{bmatrix} 7 \\ 6 \end{bmatrix}} = \underset{\hat{\mathbf{y}}}{\begin{bmatrix} 8 \\ 4 \end{bmatrix}} + \underset{(\mathbf{y} - \hat{\mathbf{y}})}{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$



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Definition

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- If W is the subspace spanned by such a set, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$ is an orthonormal basis for W.
- The standard basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is an **orthonormal basis**.

Example

Check that

$$\left\{ \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix} \right\}$$

form an orthonormal basis of $\mathbb{R}^3$.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -3/\sqrt{66} + 2/\sqrt{66} + 1/\sqrt{66} = 0$$
$$\mathbf{v}_1 \cdot \mathbf{v}_3 = -3/\sqrt{726} - 4/\sqrt{726} + 7/\sqrt{726} = 0$$
$$\mathbf{v}_2 \cdot \mathbf{v}_3 = 1/\sqrt{396} - 8/\sqrt{396} + 7/\sqrt{396} = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 9/11 + 1/11 + 1/11 = 1$$
$$\mathbf{v}_2 \cdot \mathbf{v}_2 = 1/6 + 4/6 + 1/6 = 1$$
$$\mathbf{v}_3 \cdot \mathbf{v}_3 = 1/66 + 16/66 + 49/66 = 1$$

Theorem

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

$$U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \mathbf{u}_1^T\mathbf{u}_3 \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & \mathbf{u}_2^T\mathbf{u}_3 \\ \mathbf{u}_3^T\mathbf{u}_1 & \mathbf{u}_3^T\mathbf{u}_2 & \mathbf{u}_3^T\mathbf{u}_3 \end{bmatrix}$$

Theorem

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Theorem

Let U be an $m \times n$ matrix with orthonormal columns, and let $\boldsymbol{x}$ and $\boldsymbol{y}$ be in $\mathbb{R}^n$. Then

- $\|U\boldsymbol{x}\| = \|\boldsymbol{x}\|$
- $(U\boldsymbol{x}) \cdot (U\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$
- $(U\boldsymbol{x}) \cdot (U\boldsymbol{y}) = 0$ if and only if $\boldsymbol{x} \cdot \boldsymbol{y} = 0$

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If U is a square matrix, it is called an **orthogonal matrix**.

Exercises

- Let $\boldsymbol{u}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\boldsymbol{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.
- Let $\boldsymbol{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find the orthogonal projection $\hat{\boldsymbol{y}}$ of $\boldsymbol{y}$ onto $\boldsymbol{u}$.
- Let U be an $n \times n$ orthogonal matrix. Show that $\det U = \pm 1$.