6.4 The Gram-Schmidt Process

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These slides are adapted from Linear Algebra course in UESTC

Outline

The Gram-Schmidt Process

Given a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\boldsymbol{v}_1 = \boldsymbol{x}_1$$

The Gram-Schmidt Process

Given a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{x}_1 \\ \boldsymbol{v}_2 & \end{aligned}$$

The Gram-Schmidt Process

Given a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$
\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{x}_1 \\
\boldsymbol{v}_2 &=
\end{aligned}
$$

The Gram-Schmidt Process

Given a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

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The Gram-Schmidt Process

Given a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{x}_1 \\ \boldsymbol{v}_2 &= \boldsymbol{x}_2 - \frac{\boldsymbol{x}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1 \end{aligned}$$

The Gram-Schmidt Process

Given a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\boldsymbol{v}_1 = \boldsymbol{x}_1$$

$$\boldsymbol{v}_2 = \boldsymbol{x}_2 - \frac{\boldsymbol{x}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1$$

$$\boldsymbol{v}_3$$

The Gram-Schmidt Process

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\boldsymbol{v}_3 &=
\end{aligned}
$$

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\boldsymbol{v}_3 &= \boldsymbol{x}_3
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\boldsymbol{v}_3 &= \boldsymbol{x}_3 - \frac{\boldsymbol{x}_3 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1 - \frac{\boldsymbol{x}_3 \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2} \boldsymbol{v}_2
\end{aligned}$$

The Gram-Schmidt Process

Given a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

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The Gram-Schmidt Process

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&\vdots \\
\boldsymbol{v}_p &
\end{aligned}
$$

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&\vdots \\
\boldsymbol{v}_p &=
\end{aligned}$$

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&\vdots \\
\boldsymbol{v}_p &= \boldsymbol{x}_p
\end{aligned}$$

The Gram-Schmidt Process

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&\vdots \\
\boldsymbol{v}_p &= \boldsymbol{x}_p - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 - \cdots - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_{p-1}}{\boldsymbol{v}_{p-1} \cdot \boldsymbol{v}_{p-1}}\boldsymbol{v}_{p-1}
\end{aligned}$$

The Gram-Schmidt Process

Given a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

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&\vdots \\
\boldsymbol{v}_p &= \boldsymbol{x}_p - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 - \cdots - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_{p-1}}{\boldsymbol{v}_{p-1} \cdot \boldsymbol{v}_{p-1}}\boldsymbol{v}_{p-1}
\end{aligned}
$$

Then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is an orthogonal basis for W.

The Gram-Schmidt Process

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&\vdots \\
\boldsymbol{v}_p &= \boldsymbol{x}_p - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 - \cdots - \frac{\boldsymbol{x}_p \cdot \boldsymbol{v}_{p-1}}{\boldsymbol{v}_{p-1} \cdot \boldsymbol{v}_{p-1}}\boldsymbol{v}_{p-1}
\end{aligned}$$

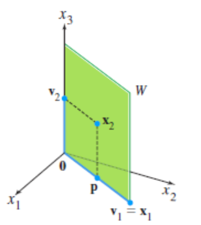
Then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is an orthogonal basis for W. And

$$\text{Span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} = \text{Span}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \quad \text{for} \quad 1 \le k \le p$$

Example

1) Let W =Span$\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$, where $\boldsymbol{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Construct an orthogonal basis for W.



$$\mathbf{v}_2 = \mathbf{x}_2 - \mathbf{p} = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

How to construct an orthonormal basis?

How to construct an orthonormal basis?

Example

Let W =Span$\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$, where $\boldsymbol{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. Construct an orthonormal basis for W.

Exercises

1) Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} -1 & 6 & 5 \\ 3 & -8 & -5 \\ 1 & -2 & -1 \\ 1 & -4 & -3 \end{bmatrix}$$

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