Sec 7.1 Diagonalization of Symmetric Matrices

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These slides are adapted from Linear Algebra course in UESTC

Definition

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Are these matrices symmetric?

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & -2 \\ -1 & 2 & 3 \\ -2 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & -2 \\ 1 & 2 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

Example

If possible, diagonalize the matrix

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

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- Characteristic equation
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- Characteristic equation
  $-\lambda^3 + 17\lambda^2 - 90\lambda + 144 = -(\lambda - 8)(\lambda - 6)(\lambda - 3) = 0$
- $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Theorem

If A is symmetric, then any two eigenvectors from different eigenspace are **orthogonal**.

Orthogonally Diagonalizable

An $n \times n$ matrix A is said to be **orthogonally diagonalizable** if there are an orthogonal matrix P and a diagonal matrix D such that

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Theorem

An $n \times n$ matrix A is **orthogonally diagonalizable** if and only if A is a symmetric matrix.

Example

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix},$$

whose characteristic polynomial is

$$\lambda^3 - 12\lambda^2 + 21\lambda + 98 =$$

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$$\lambda^3 - 12\lambda^2 + 21\lambda + 98 = (\lambda - 7)^2(\lambda + 2)$$

$$\lambda = 7: \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \qquad \lambda = -2: \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$\mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

$$P = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3\,] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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- The eigenspaces are mutually orthogonal.
- A is orthogonally diagonalizable.

Spectral Decomposition

$$A = PDP^T$$

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$$A = PDP^T = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}$$

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\begin{aligned}
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&= \begin{bmatrix} \lambda_1 u_1 & \cdots & \lambda_n u_n \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}
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Then we have

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$$

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$$

Exercises

- Orthogonally diagonalize this matrix, provided that its eigenvalues are -2 and 7

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

- Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that $B^T AB$, $B^T B$ and BB^T are symmetric matrices.
- Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.