1.2 Row Reduction and Echelon Forms

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These slides are adapted from Linear Algebra course in UESTC

Outline

A few definitions:

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- **nonzero** row (or column): a row or column that contains at least one nonzero entry;

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- **nonzero** row (or column): a row or column that contains at least one nonzero entry;
- **leading entry**: the leftmost nonzero entry (in a nonzero row)

Echelon Form (REF)

Echelon form (row echelon form)

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2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

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4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Echelon Form

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

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Reduced Echelon Form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

A matrix in (reduced) echelon form is called a **(reduced) echelon matrix**.

Echelon Form

Are the following matrix in (reduced) echelon form?

1. $$\begin{bmatrix} 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. $$\begin{bmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall: row equivalent

Three elementary row operations

1. **Replacement** Replace one row by the sum of itself and a multiple of another row.
2. **Interchange** Interchange two rows.
3. **Scaling** Multiply a row by a nonzero constant.

If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form** (or row echelon form) of A; if U is in reduced echelon form, we call U **the reduced echelon form** of A.

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Uniqueness of the **RREF**

Each matrix is row equivalent to one and **only one** reduced echelon matrix.

Pivot Position and Pivot Column

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

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Example

Reduce the matrix to REF, then RREF

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

The Row Reduction Algorithm

1. Begin with the **leftmost nonzero column**. This is a pivot column. The pivot position is at the top.
2. Select a **nonzero entry in the pivot column as a pivot**. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to **create zeros** in all positions below the pivot.

The Row Reduction Algorithm

4. Repeat the process until there are no more nonzero rows to modify.

5. Beginning with the **rightmost pivot** and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Example

Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & 11 & 8 & -5 & 8 & 9 \\ 3 & 9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solutions of Linear Systems

Important facts

- A linear system corresponds to **a unique augmented matrix** and vise versa.

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- A linear system corresponds to **a unique augmented matrix** and vise versa.
- Row reduction **does not change** the solution set.

Solutions of Linear Systems

Example

The equivalent RREF of the augmented matrix of a linear system $\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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- **basic variable**: the variables corresponding to pivot columns in the matrix.

Solutions of Linear Systems

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The equivalent RREF of the augmented matrix of a linear system $\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- **basic variable**: the variables corresponding to pivot columns in the matrix.
- **free variable**: the other variables.

Existence and Uniqueness Theorem

A linear system is consistent **if and only if** an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \qquad \text{with } b \text{ nonzero}$$

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If a linear system is consistent, then the solution set contains either

1. a unique solution, when there are no **free variables**, or
2. infinitely many solutions, when there is at least one free variable

Example

Find the general solution of the linear system whose augmented matrix is

1. $\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 1 & 1 & -4 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 1 & 6 & 4 & -13 & -2 & 6 \end{bmatrix}$