1.3 Vector Equations

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These slides are adapted from Linear Algebra course in UESTC

Outline

1. Vectors in $\mathbb{R}^2$
2. Geometric Description of $\mathbb{R}^2$
3. Vectors in $\mathbb{R}^3$, $\mathbb{R}^n$
4. Linear Combinations
5. Span

Vectors in $\mathbb{R}^2$

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- $\mathbb{R}^2$: the set of all vectors with two entries.

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- Sum: their **sum** is the vector $u + v$ obtained by adding corresponding entries of u and v.
- Scalar multiple: given a real number c, the **scalar multiple** of u and c is the vector cu obtained by multiplying each entry in u by c.

Linear Operations

Example

Given $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, find $4u$, $(-2)v$, and $4u + (-2)v$.

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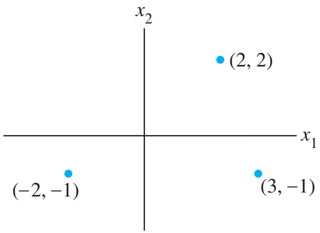

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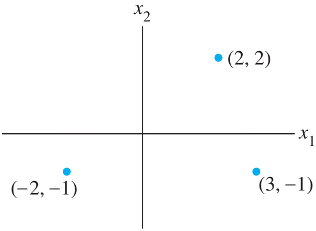

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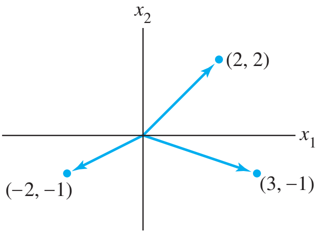

(b) Vectors with arrows

Parallelogram Rule for Addition

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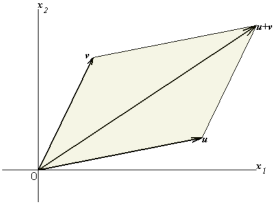
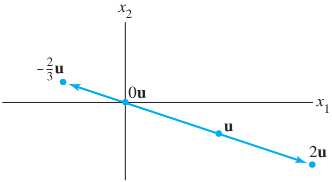
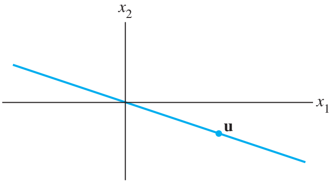


Multiple of a Vector



Typical multiples of **u**



The set of all multiples of **u**

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- Sum and scalar multiple?

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Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

Linear Combinations of Vectors

Definition of Linear Combination

Given a set of vectors $v_1, v_2, \cdots, v_k$, where k is an integer, then

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k$$

is a **linear combination** of vectors $v_1, v_2, \cdots, v_k$, where $x_1, x_2, \cdots, x_k \in \mathbb{R}$.

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Example

$$2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0.5 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

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$$2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0.5 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Example

Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$.

Determine whether b can be generated (or written) as a linear combination of a_1 and a_2.

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If this vector equation has a solution, find it.

Vector Equation

A vector equation

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- In particular, b can be generated by a linear combination of $a_1, a_2, \ldots, a_n$ if and only if there exists a solution to ...

An Important Note

A Key Idea

One of the key ideas in linear algebra is to study the set of all vectors taht can be generated or written as a linear combination of a fixed set $\{v_1, \ldots, v_p\}$ of vectors.

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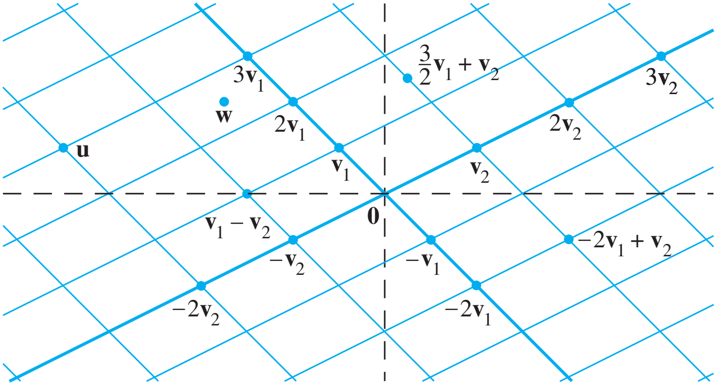
Example

Consider possible linear combinations of $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



3/2 v1 + v2
3v1
w
2v1
3v2
2v2
u
v1
v2
0
v1 − v2
−v2
−v1
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Span

Definition

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$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1, \ldots, c_p$ scalars.

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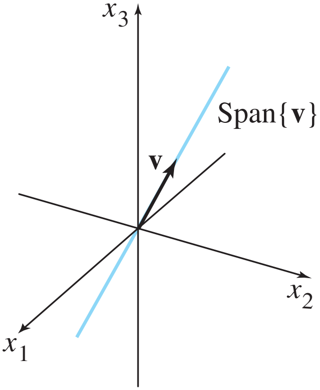

(a) Span{v} as a line through the origin

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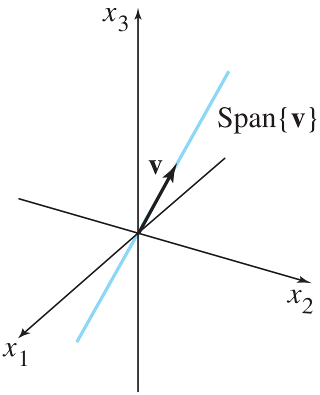

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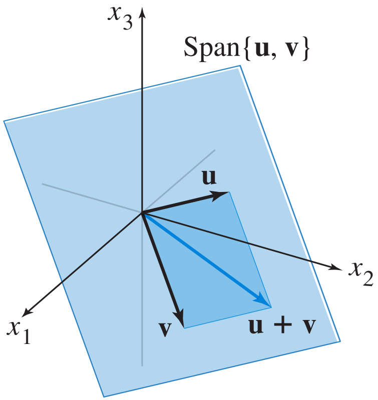

(b) Span{u,v} as a plane through u, v and the origin

Example

Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$.

Then Span$\{a_1, a_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is b in that plane?

In-Class Practice

For what value(s) of h will $\mathbf{y}$ be in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

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Question

- Geometrically, what is Span$\{v_1, v_2, v_3\}$?
- Is Span$\{u, v\}$ always visualized as a plane through the origin.