1.4 The Matrix Equation $Ax = b$

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These slides are adapted from Linear Algebra course in UESTC

OUTLINE

1. Matrix-Vector Product
2. Existence Solutions
3. Computation of Ax
4. Properties of the Matrix-Vector Product Ax

Matrix-Vector Product

Definition

If A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if x is in $\mathbb{R}^n$, then the product of A and x, denoted by Ax, is the linear combination of the columns of A using the corresponding entries in x as weights;

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$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + \cdots + x_n a_n$$

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If the number of entries in x is not equal to the number of columns, Ax is **not defined**.

Example

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

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Example

For $v_1, v_2, v_3 \in \mathbb{R}^m$, write the linear combination $3v_1 - 5v_2 + 7v_3$ as a matrix times a vector.

Matrix Equation

If A is an $m \times n$ matrix, with columns $a_1, \dots, a_n$, and if b is in $\mathbb{R}^m$, the matrix equation

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which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

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- Is b in Span$\{a_1, \ldots, a_n\}$?
- Is $Ax = b$ consistent?
- Whether the equation $Ax = b$ is consistent for all possible b?

Existence of Solutions

Example

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible b_1, b_2, b_3?

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

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Warning: A is the coefficient matrix.

Computation of Ax

Example

Compute Ax, where $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

So many calculation!

Example

Compute the following matrix-vector products.

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

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The last matrix is called an identity matrix and is denoted by I.

Properties of the Matrix-Vector Product

Theorem

If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then

1. A(u+v) = Au + Av
2. A(cu) = c(Au)

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Example

Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $v = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify this theorem by computing $A(u+v)$ and $Au + Av$.

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$

- How many rows of A contain a pivot position? Does the equation $Ax = b$ have a solution for each b in $\mathbb{R}^4$?
- Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of A? Does the columns of A span $\mathbb{R}^4$?