1.4 The Matrix Equation Ax = b

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These slides are adapted from Linear

Algebra course in UESTC

OUTLINE

- Matrix-Vector Product
- Existence Solutions
- **3** Computation of Ax
- Properties of the Matrix-Vector Product Ax

Definition

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If the number of entries in x is not equal to the number of columns, Ax is **not defined**

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

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Example

For $v_1, v_2, v_3 \in \mathbb{R}^m$, write the linear combination $3v_1 - 5v_2 + 7v_3$ as a matrix times a vector.

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which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

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- Is Ax = b consistent?
- Whether the equation Ax = b is consistent for all possible b?

Example

Let
$$A=\begin{bmatrix}1&3&4\\-4&2&-6\\-3&-2&-7\end{bmatrix}$$
 and $b=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$. Is the

equation $\bar{A}x = b$ consistent for all possible b_1, b_2, b_3 ?

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

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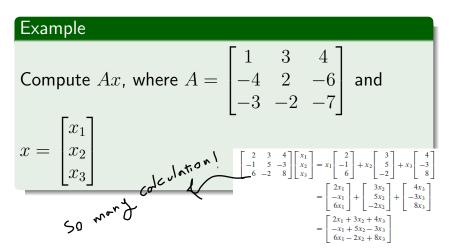
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Warning: A is the coefficient matrix.



Computation of Ax



Compute the following matrix-vector products.

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

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The last matrix is called an identity matrix and is denoted by I.



Properties of the Matrix-Vector Product

Theorem

If A is an $m \times n$ matrix, u and v are vectors in \mathbb{R}^n , and c is a scalar, then

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Example

Let
$$A=\begin{bmatrix}2&5\\3&1\end{bmatrix}$$
, $u=\begin{bmatrix}4\\-1\end{bmatrix}$, and $v=\begin{bmatrix}-3\\5\end{bmatrix}$. Verify this theorem by computing $A(u+v)$ and $Au+Av$.

Let
$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$$

- How many rows of A contain a pivot position? Does the equation Ax = b have a solution for each b in \mathbb{R}^4 ?
- Can each vector in \mathbb{R}^4 be written as a linear combination of the columns of A? Does the columns of A span \mathbb{R}^4 ?