1.5 Solution Sets of Linear Systems

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These slides are adapted from Linear Algebra course in UESTC

Outline

Homogeneous Linear Systems

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$$Ax = 0$$

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Homogeneous Linear Systems

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Theorem

The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one **free variable**.

Homogeneous Linear Systems

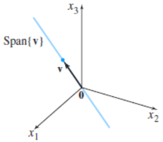
Example

Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= 0 \\ x_2 \qquad &= 0 \\ 0 &= 0 \end{aligned}$$



$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous Linear Systems

Example

A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous "system"

$$10x_1 - 3x_2 - 2x_3 = 0$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \underset{\substack{\uparrow \\ \mathbf{u}}}{\begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix}} + x_3 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}} \quad \text{(with } x_2, x_3 \text{ free)}$$

Homogeneous Linear Systems

$$x = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} \qquad x = sv + tu$$

(with x_2, x_3 free) $\qquad (s, t \in \mathbb{R})$

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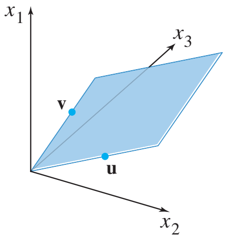

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- Solving this equation amounts to finding an explicit description of the plane as the set spanned by u and v.
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$$x = su + tv \quad (s, t \in \mathbb{R})$$

The solution is in parametric vector form.

Nonhomogeneous Linear Systems

Example

Describe all solutions of $Ax = b$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 \quad - \tfrac{4}{3}x_3 &= -1 \\ x_2 \qquad &= 2 \\ 0 &= 0 \end{aligned}$$

Nonhomogeneous Linear Systems

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

$$x = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

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$$(t \in \mathbb{R})$$

Nonhomogeneous Linear Systems

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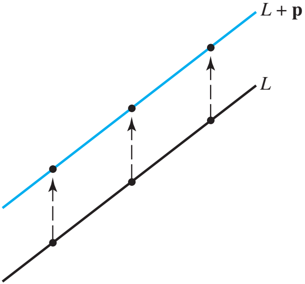
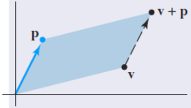
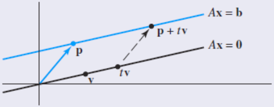
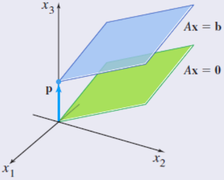


Geometric understanding







Nonhomogeneous Linear Systems

Theorem

Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogeneous equation $Ax = 0$.

Writing a solution set in parametric vector form

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1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution x as a vector whose entries depend on the free variables, if any.
4. Decompose x into a linear combination of vectors using the free variables as parameters.

Exercise

Describe the solutions of the following linear systems in parametric vector form. And give a geometric description of the solution set.

$$\begin{aligned} x_1 \quad +3x_2 \quad +x_3 &= 0 \\ -4x_1 \quad -9x_2 \quad +2x_3 &= 0 \\ -x_2 \quad -6x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 \quad +3x_2 \quad +x_3 &= 1 \\ -4x_1 \quad -9x_2 \quad +2x_3 &= -1 \\ -x_2 \quad -6x_3 &= -3 \end{aligned}$$

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (2)$$

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ free variable

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \qquad - 2x_3 &= 0 \\ x_2 + \ x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Mark each statement true or false

Every matrix is row equivalent to a unique matrix in echelon form.

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If matrices A and B are row equivalent, they have the same reduced echelon form.

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If A is am $m \times n$ matrix and the equation $Ax = b$ is consistent for every b in $\mathbb{R}^m$, then A has m pivot positions.