2.1 Matrix Operations

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These slides are adapted from Linear Algebra course in UESTC

Outline

1. Linear Operations (Sums and Scalar Multiples)
2. Matrix Multiplication
3. Transpose of a Matrix

An $m \times n$ matrix

An $m \times n$ matrix

Column j

Row i

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A$$

$$\begin{matrix} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{matrix}$$

An $m \times n$ matrix

$$\text{Row } i \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A$$

Column j

$\uparrow$ $\mathbf{a}_1$ $\quad$ $\uparrow$ $\mathbf{a}_j$ $\quad$ $\uparrow$ $\mathbf{a}_n$

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

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- Diagonal entries
- Diagonal matrix, square matrix

An $m \times n$ matrix

$$
\begin{array}{cc}
 & \text{Column } j \\
\text{Row } i &
\begin{bmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
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\end{array}
$$

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

- Diagonal entries
- Diagonal matrix, square matrix
- Zero matrix 0

Linear Operations

Two matrices are **equal** if they have the same size and if their corresponding columns are equal.

Sums and Scalar Multiples

If A and B are $m \times n$ matrices, r a scalar

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- **Scalar Multiple**

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- **Sum** $A + B = [a_1 + b_1, \cdots, a_n + b_n]$
- **Scalar Multiple** $rA = [ra_1, \cdots, ra_n]$

Linear Operations

Example

Let

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 1 & 2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Properties of Linear Operations

Theorem

Let A, B and C be matrices of the same size, and let r and s be scalars.

$a)$. $A + B = B + A$

$b)$. $(A + B) + C = A + (B + C)$

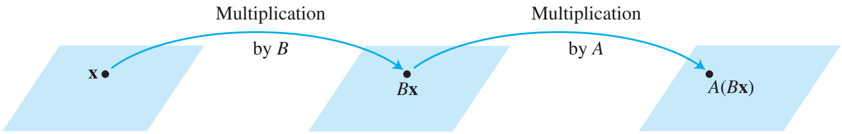
$c)$. $A + 0 = A$

$d)$. $r(A + B) = rA + rB$

$e)$. $(r + s)A = rA + sA$

$f)$. $r(sA) = (rs)A$

Matrix Multiplication



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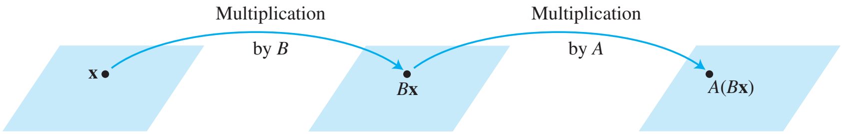

$$\forall x, A(Bx) = (AB)x$$

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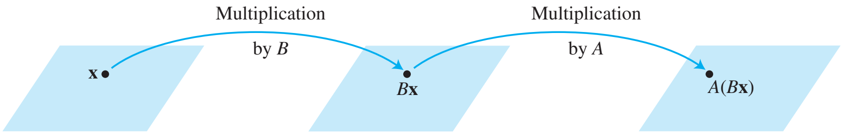

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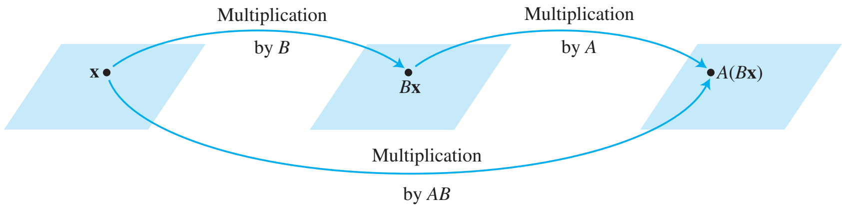

Definition

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \cdots, b_p$, then the product AB is the $m \times p$ matrix whose columns are $Ab_1, \cdots, Ab_p$. That is,

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- Multiplication of matrices corresponds to composition of linear transformations.

Example

Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and

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Then

$$AB = A[\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,] = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

$A\mathbf{b}_1$ $A\mathbf{b}_2$ $A\mathbf{b}_3$

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

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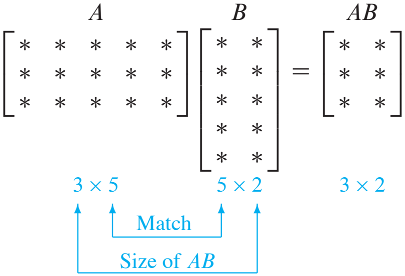
- AB has the same number of rows as A and the same number of columns as B.

Example

If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?



Exercise

Calculate AB with

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

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$$\text{row}_i(AB) = \text{row}_i(A)B$$

Properties of Matrix Multiplication

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5. $I_m A = A = AI_N$

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1. In general, $AB \neq BA$. If $AB = BA$, we say they **commute** with each other.
2. The cancellation laws do not hold for matrix multiplication. $AB = AC$ does not in general implies $B = C$.
3. $AB = 0$, you cannot conclude in general that either $A = 0$ or $B = 0$.

Powers and Transpose of a Matrix

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$$A^k = A \cdots A$$

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Transpose of a Matrix

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Example

Write down the transposes of the following matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

Properties of the Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- For any scalar r, $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$

Mark each statement True or False. Justify each answer.

- The second row of AB is the second row of A multiplied on the right by B.
- $(AB)C = (AC)B$
- $(AB)^T = A^T B^T$
- $A^T + B^T = (A + B)^T$